Quasiorder lattices of varieties

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Novi Sad, 2015. June 5–7.

Definition

The set of **compatible quasiorders** of an algebra $\mathbf{A}$ is

$$\mathrm{Quo}(\mathbf{A}) = \{\, \alpha \leq \mathbf{A}^2 \mid \alpha \text{ is reflexive and transitive} \,\}.$$

1. A quasiorder $\alpha \subseteq A^2$ is compatible with $\mathbf{A}$ if
$$(x, y) \in \alpha \implies (p(x), p(y)) \in \alpha$$
for all unary polinomials p of $\mathbf{A}$.
2. $\mathrm{Quo}(\mathbf{A})$ forms an (involution) lattice with $\alpha \wedge \beta = \alpha \cap \beta$ and $\alpha \vee \beta = \overline{\alpha \cup \beta}$, where $\overline{\alpha \cup \beta}$ is the transitive closure of $\alpha \cup \beta$.
3. The set $\mathrm{Con}(\mathbf{A})$ of congruences forms a sublattice of $\mathrm{Quo}(\mathbf{A})$.

Goal

Systematic study of the connection between congruence identities, quasiorder identities and Maltsev conditions satisfied by varieties.

Why study compatible quasiorders?

1. More general than congruences.
2. Better behaved than tolerances.
3. Some connection with the constraint satisfaction problem:

For a subdirect power $\mathbf{R} \leq_{\mathrm{sd}} \mathbf{A}^n$ and a closed path

$$p := k_1 \to k_2 \to \cdots \to k_m \to k_1 \quad \text{with} \quad k_i \in \{1, \ldots, n\}$$

define

$$\alpha_p = \bigcup_{i=1}^{\infty} (\eta_{k_1} \circ \eta_{k_2} \circ \cdots \circ \eta_{k_m})^i \quad \text{where} \quad \eta_k = \ker \pi_k.$$

We have $\alpha_p \in \mathrm{Quo}(\mathbf{R})$ and $\alpha_p \vee \eta_{k_1}$ can be computed from the following two-projections:

$$\pi_{k_1 k_2}(R),\ \pi_{k_2 k_3}(R), \ldots, \pi_{k_m k_1}(R).$$

"Prague strategy" iff $\mathrm{range}(p) \subseteq \mathrm{range}(q) \implies \alpha_p \leq \alpha_q$.

Is this study interesting?

Main results:

1. A locally finite variety $\mathcal{V}$ is congruence distributive ($\mathrm{Con}(\mathbf{A})$ is distributive for all $\mathbf{A} \in \mathcal{V}$) if and only if it is quasiorder distributive ($\mathrm{Quo}(\mathbf{A})$ is distributive for all $\mathbf{A} \in \mathcal{V}$).
2. A locally finite variety is congruence modular if and only if it is quasiorder modular.
3. The variety of semilattices is not quasiorder meet semi-distributive (but it is congruence meet semi-distributive).
4. $\mathrm{Quo}(\mathbf{A})$ is not in the lattice quasivariety generated by the congruence lattices $\mathrm{Con}(\mathbf{B})$ for $\mathbf{B} \in \mathrm{HSP}(\mathbf{A})$.
5. For a finite algebra $\mathbf{A}$ in a congruence meet semi-distributive variety $\mathrm{Quo}(\mathbf{A})$ has no sublattice isomorphic to $\mathbf{M}_3$.
6. We conjecture/show that there is an infinite semilattice whose quasiorder lattice contains a sublattice isomorphic to $\mathbf{M}_3$.

Congruence distributivity

Theorem (B. Jónsson, 1967)

A variety is congruence distributive iff it has Jónsson terms

$$x \approx p_1(x,x,y) \text{ and } p_n(x,y,y) \approx y,$$
$$p_i(x,y,y) \approx p_{i+1}(x,y,y) \text{ for odd } i,$$
$$p_i(x,x,y) \approx p_{i+1}(x,x,y) \text{ for even } i, \text{ and}$$
$$p_i(x,y,x) \approx x \text{ for all } i.$$

Theorem (G. Czédli and A. Lenkehegyi, 1983; I. Chajda, 1991)

There is a Maltsev condition charaterizing quasiorder distributivity.

Corollary (G. Czédli and A. Lenkehegyi, 1983)

If a variety $\mathcal{V}$ has a majority term, then it is quasiorder distributive.

Directed Jónsson terms

Definition

The ternary terms $p_1, \dots, p_n$ are **directed Jónsson terms** if

$$x \approx p_1(x,x,y) \text{ and } p_n(x,y,y) \approx y,$$
$$p_i(x,y,y) \approx p_{i+1}(x,x,y) \text{ for } i = 1, \dots, n-1, \text{ and}$$
$$p_i(x,y,x) \approx x \text{ for } i = 1, \dots, n.$$

Theorem (A. Kazda, M. Kozik, R. McKenzie and M. Moore, 2014)

A variety is congruence distributive if and only if it has directed Jónsson terms.

Lemma (A. Kazda, M. Kozik, R. McKenzie and M. Moore, 2014)

If $\alpha \vartriangleleft_{\mathrm{WJ}} \beta$ (weak Jónsson absorbs) for $\alpha, \beta \in \mathrm{Quo}(\mathbf{A})$ then $\alpha = \beta$.

Theorem (L. Barto, 2012)

Finitely related algebras in congruence distributive varieties have near unanimity terms.

$$t(y,x,\dots,x) \approx t(x,y,x\dots,x) \approx \dots \approx t(x,\dots,x,y) \approx x.$$

Theorem

A locally finite variety is congruence distributive if and only if it has directed Jónsson terms.

Proof.

Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y)$ be the two-generated free algebra, and put

$$R = \mathrm{Sg}\{(x,x,x),(x,y,y),(y,x,y)\} \leq \mathbf{F}^3.$$

The algebra $(F;\mathrm{Pol}(R))$ is finitely related and has Jónsson terms, so R has a near-unanimity polymorphism t. The terms generating the tuples $t((y,x,y),\dots,(y,x,y),(x,y,y),(x,x,x),\dots,(x,x,x))$ are directed Jónsson terms. □

Theorem

If a finite algebra has directed Jónsson terms, then it is quasiorder distributive.

Proof.

1. We show $(\alpha \vee \beta) \wedge \gamma \leq (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$ for $\alpha, \beta, \gamma \in \mathrm{Quo}(\mathbf{A})$
2. Put $\gamma^* = \gamma \cap \gamma^{-1} \in \mathrm{Con}(\mathbf{A})$
3. Choose $(a, b) \in (\alpha \vee \beta) \wedge \gamma - (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$ such that the interval $[a/\gamma^*, b/\gamma^*]$ is minimal in the poset $(A/\gamma^*; \gamma/\gamma^*)$
4. We have a chain of $\alpha \cup \beta$ links connecteing a and b
5. Use the directed Jónsson terms to move this chain inside the interval $[a, b] = \{ x \mid a \, \gamma \, x \, \gamma \, b \}$.
6. The links inside a/γ^* are in $(\alpha \wedge \gamma) \cup (\beta \wedge \gamma)$.
7. The first link leaving a/γ^* is also in $(\alpha \wedge \gamma) \cup (\beta \wedge \gamma)$.
8. By minimality the rest is also in $(\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$. □

Theorem

For a locally finite variety $\mathcal{V}$ the following are equivalent:

1. $\mathcal{V}$ *is congruence distributive,*
2. $\mathcal{V}$ *has [directed] Jónsson terms,*
3. $\mathcal{V}$ *is quasiorder distributive.*

Problem

Does the above equivalence hold for all varieties? Does quasiorder distributivity imply directed Jónsson terms syntactically?

Theorem

For a finite algebra with directed Jónsson terms and α, β *compatible reflexive relations we have* $\overline{\alpha} \cap \overline{\beta} = \overline{\alpha \cap \overline{\beta}}$.

Problem

Do we have $\overline{\alpha} \cap \overline{\beta} = \overline{\alpha \cap \beta}$ in the above theorem? Is taking the transitive closure a lattice homomorphism (for monounary algs)?

Directed Gumm terms

Definition

The ternary terms $p_1, \ldots, p_n, q$ are **directed Gumm terms** if

$$\begin{aligned} x &\approx p_1(x,x,y), \\ p_i(x,y,y) &\approx p_{i+1}(x,x,y) \text{ for } i = 1,\ldots,n-1, \\ p_i(x,y,x) &\approx x \text{ for } i = 1,\ldots,n, \\ p_n(x,y,y) &\approx q(x,y,y) \text{ and } q(x,x,y) \approx y. \end{aligned}$$

Theorem (A. Kazda, M. Kozik, R. McKenzie and M. Moore, 2014)

A variety is congruence modular if and only if it has directed Gumm terms.

- Has been known for locally finite varieties (M. Kozik)
- Similar trick works to show this (L. Barto: finitely related algebras in congruence modular varietes have edge term)

Congruence modularity

Theorem

If a finite algebra has directed Gumm terms then the lattice of its compatible quasiorders is modular.

- To show $\alpha \leq \gamma \implies (\alpha \vee \beta) \wedge \gamma \leq \alpha \vee (\beta \wedge \gamma)$ we take again a counterexample pair (a, b) with minial distance in γ/γ^*.
- Significantly harder than the distributive case.

Theorem

For a locally finite variety $\mathcal{V}$ the following are equivalent:

1. *$\mathcal{V}$ is congruence modular,*
2. *$\mathcal{V}$ has [directed] Gumm terms,*
3. *$\mathcal{V}$ is quasiorder modular.*

Proposition (I. Chajda, 1991)

In n-permutable varieties compatible quasiorders are congruences.

Definition

A variety is **congruence meet semi-distributive** if the congruence lattices of its algebras satisfy

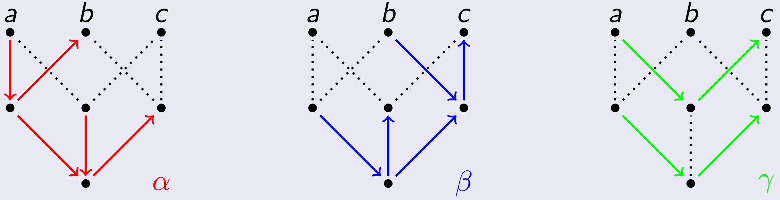
$$\alpha \wedge \gamma = \beta \wedge \gamma \implies (\alpha \vee \beta) \wedge \gamma = \alpha \wedge \gamma.$$

The dual condition is **congruence join semi-distributivity**.

Typical meet semi-distributive variety is the variety of semilattices (or varieties with totally symmetric operations of all arities).

Proposition

The variety of semilattices is not quasiorder meet semi-distributive.



Theorem (D. Hobby and R. McKenzie, TCT Theorem 9.10)

For any locally finite variety $\mathcal{V}$ the following are equivalent:

1. $\operatorname{typ}\{\mathcal{V}\} \cap \{\mathbf{1}, \mathbf{2}\} = \emptyset$.
2. $\mathcal{V}$ *satisfies an idempotent linear Maltsev condition that does not hold in the varieties of vectorspaces over finite fields.*
3. $\mathcal{V} \models_{\mathrm{CON}} \gamma \wedge (\alpha \circ \beta) \subseteq \alpha_m \wedge \beta_m$ *for some* m *where* $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_{n+1} = \alpha \vee (\gamma \wedge \beta_n)$ *and* $\beta_{n+1} = \beta \vee (\gamma \wedge \alpha_n)$.
4. $\mathbf{M}_3$ *is not a sublattice of* $\mathrm{Con}(\mathbf{A})$ *for any* $\mathbf{A} \in \mathcal{V}$.
5. $\mathcal{V}$ *is congruence meet semi-distributive.*
6. *There are no non-trivial abelian congruences.*

- The previous example shows that $\mathbf{D}_1$ is a sublattice of the quasiorder lattice of the free semilattice with three generators.
- So items (3) and (5) do not hold for quasiorder lattices.

Minimal algebras

Definition

A finite algebra **A** is (α, β)-**minimal** for $\alpha, \beta \in \mathrm{Quo}(\mathbf{A})$ with $\alpha < \beta$ if every unary polynomial is either a permutation or $p(\beta) \subseteq \alpha$.

The very beginning of tame congruence theory (excluding the classification of minimal algebras) goes through.

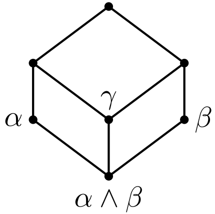
Proposition (c.f. D. Hobby and R. McKenize, TCT Theorem 2.8)

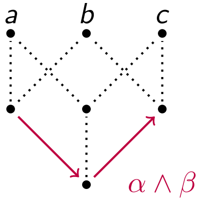
*Let (α, β) be a tame quasiorder quotient of a finite algebra **A**. Then all (α, β)-minimal sets of **A** are polynomially isomorphic.*

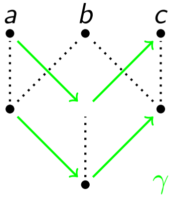
Proposition (c.f. D. Hobby and R. McKenize, TCT Lemma 2.10)

*Let **A** be a finite algebra and $\alpha < \beta$ be quasiorders of **A** such that the interval lattice $[\alpha, \beta]$ in $\mathrm{Quo}(\mathbf{A})$ has no strictly increasing, non-constant, meet edomorphism. Then every (α, β)-minimal set is the range of an idempotent unary polynomial.*

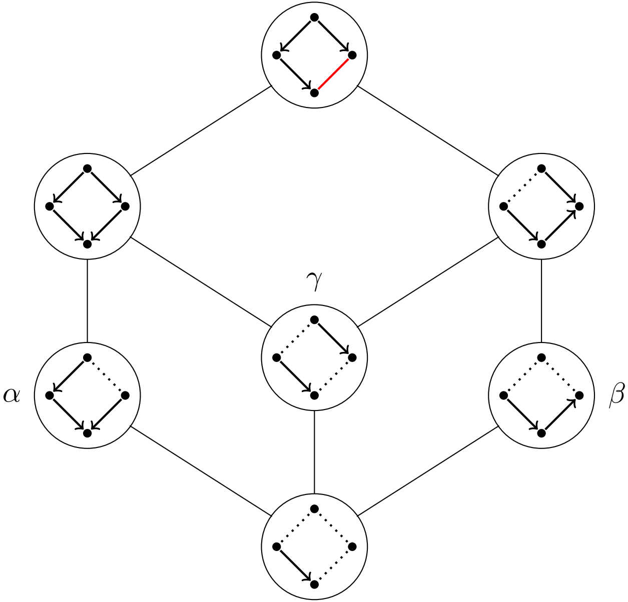
- Consider again the quasiorder lattice of the free semilattce with three generators **S**, which has a sublattice isomorphic to $\mathbf{D}_1$.







- $\mathbf{D}_1$ has critical quotient $(\alpha \wedge \beta, \gamma)$, corresponding to meet semi-distributivity.
- We can take the image of **S** under the idempotent polynomial $p(x) = a \wedge x$.
- We have $p(\gamma) \not\subseteq \alpha \wedge \beta$ so p embeds the $\mathbf{D}_1$ sublattice into the quasiorder lattice of $p(\mathbf{A})$.


γ
α
β

Theorem

For a finite algebra $\mathbf{A}$ *in a congruence meet semi-distributive variety* $\mathrm{Quo}(\mathbf{A})$ *does not have a sublattice isomorphic to* $\mathbf{M}_3$*.*

Proof.

1. Choose a minimal sublattice of $\mathrm{Quo}(\mathbf{A})$ isomorphic to $\mathbf{M}_3$.
2. The botton quasiorder α cannot have a double edge.
3. The top quasiorder β must have a double edge.
4. The top quasiorder β must be a congruence.
5. The algebra must be (α, β)-minimal.
6. The algebra must be $(0, \beta)$-minimal.
7. We are back to congruences, use classification of minimal algebras. □

Work in progress

- We (hope to) have a construction of an infinite semilattice whose lattice of compatible quasiorders has an $\mathbf{M}_3$ sublattice.
- Working on congruence join semi-distributivity and omitting the $\mathbf{M}_3$ and $\mathbf{D}_2$ sublattices.
- Trying to find a good notion of the commutator for quasiorders (if there is such a thing).
- Interesting theorems in the TCT book (e.g. Theorem 5.26) about orderable tame quotients (types **4** and **5**) and (α, β)-preorders.

Thank You!